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Propagation of a shock wave in relativistic magnetofluids

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Abstract. The existence of a shock wave propagating in relativistic magnetofluids is assumed and the shock strength is determined. The jump discontinuities of the flow and field parameters across the shock wave are explicitly expressed in terms of parameters defined on the shock surface itself and the flow variables just on the upstream side of the shock. The flow gradients at the rear of the shock have been determined in terms of the flow parameters just ahead of the shock and their interior derivatives along the shock surface itself. The expressions for vorticity and current density generated by a hydromagnetic shock propagating in relativistic magnetofluids have been obtained. A few results of astrophysical interest have also been derived.

1. Introduction

Unless one is interested in very weak signals which occur in detection problems, either physical circumstances (the dense matter of certain astrophysical objects (Ruderman 1972), the large velocities involved in galactic motions, the effects of strong magnetic fields (Ruderman 1972)) or the very nonlinearity of the field equations necessitates the study of nonlinear wave propagation in relativistic continuous matter. We deem our analysis important for the interpretation of phenomena connected with some astrophysical objects such as neutron stars, collapsed stars etc as they possess very strong magnetic fields of intensity ($\geqslant 10^{10}$ G) frozen into the matter, and very high electrical conductivity (Lichnerowicz 1970). Relativistic magnetohydrodynamical shock waves appear in the physics of the sun, the solar system and also the galaxies (Lichnerowicz 1970).

A considerable amount of work has been done on nonlinear wave propagation in various models of relativistic continua. Taub (1948) presented theoretical foundations of relativistic shock waves in a perfect fluid model. Hoffman and Teller (1950) gave an elegant relativistic treatment of magnetohydrodynamical shock waves. Lichnerowicz (1970, 1967, 1975), Saini (1961, 1976) and many others obtained general shock relations in relativistic magnetofluids.

The problem of determining the differential effects of shock fronts on the flow variables has drawn the attention of several researchers striving for increasing generality. Thomas (1947) solved this problem for plane shocks in non-relativistic and non-conducting gases, and his results were extended by Kanwal (1958) for three-dimensional shocks in unsteady flows of ordinary gases. Ram and Mishra (1966) further generalised their results for hydromagnetic shocks in three-dimensional pseudo-stationary flows. Pant and Mishra (1965) solved this problem in the case of stationary

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flows of conducting gases and obtained the expression for vorticity generated by the shock under the restriction that the magnetic field acts tangentially to the shock surface. Ram (1968) studied this problem in the case of unsteady flows of conducting fluids with no restriction on the magnetic field, and has drawn an interesting conclusion that the vorticity and current density generated by an oblique hydromagnetic shock in three-dimensional unsteady flows depend upon the dynamical as well as thermodynamical behaviour of the fluid. However, this problem does not appear to have been solved in the relativistic framework. The main academic interest of the present paper is to determine the jump discontinuities across a relativistic magnetohydrodynamic shock front and its differential effects on the flow and field variables, and to obtain expressions for the vorticity and current density generated by the shock.

2. Basic preliminaries

Let V_4 be an Einstein–Riemann space characterised by four coordinates

$$x^\alpha = (x^k, x^4), \qquad x^4 = ct$$

whose metric ds^2, with signature $(+++-)$, is expressible in the form

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta,$$

where $g_{\alpha\beta}$ is the metric tensor of the space connected with the matter distribution in space–time through Einstein's field equations, t is the time and c is the velocity of light in vacuum. With the help of the world velocity U^α such that

$$g_{\alpha\beta} U^\alpha U^\beta = -1,$$

we define the invariant derivative $D_U \equiv U^\alpha \nabla_\alpha$ where ∇_α represents the operator of covariant derivative. We can also define the field of spatial projectors by

$$S^{\alpha\beta} = g^{\alpha\beta} + U^\alpha U^\beta$$

such that

$$S^{\alpha\beta} U_\beta = 0 \qquad \text{and} \qquad S_\alpha^{;\alpha} = 3.$$

Here and in what follows the range of Latin indices is 1, 2, 3 and that of Greek indices is 1, 2, 3, 4. A repeated index will usually imply summation unless specified otherwise.

Within a material medium, a general electromagnetic field is represented by two skew-symmetric field tensors $H_{\alpha\beta}$ and $G_{\alpha\beta}$ which satisfy the Maxwell equations

$$\nabla_\alpha \overset{*}{H}{}^{\alpha\beta} = 0 \qquad \text{and} \qquad \nabla_\alpha G^{\alpha\beta} = J^\beta,$$

where $H_{\alpha\beta}$ is the electric field–magnetic induction tensor, $G_{\alpha\beta}$ the magnetic field–electric induction tensor, $\overset{*}{H}{}^{\alpha\beta}$ the dual tensor and J^α the charge current four-vector. The spacelike four-vectors

$$e_\beta = U^\alpha H_{\alpha\beta}, \qquad b_\beta = U^\alpha \overset{*}{H}_{\alpha\beta}$$

are the electric field and the magnetic induction with respect to the timelike direction U^α and $e_\beta U^\beta = b_\beta U^\beta = 0$. If μ is the constant magnetic permeability of the fluid, $b_\beta = \mu h_\beta$, where h^α is the magnetic field vector. In a relativistic formulation, Ohm's law may be written as

$$J^\alpha = \varepsilon U^\alpha + \lambda e^\alpha$$

where ε is the charge density and λ the conductivity of the fluid. If we now assume that λe^α is finite, then for $\lambda = \infty$ we have necessarily $e^\alpha = 0$ so that the electromagnetic field is reduced to the magnetic field with respect to the fluid. Furthermore, under the assumptions of infinite electrical conductivity and constant magnetic permeability, the Maxwell equations are reduced to

$$\nabla_\alpha(U^\alpha h^\beta - U^\beta h^\alpha) = 0$$

and the total energy-momentum tensor $T^{\alpha\beta}$ of the fluid and electromagnetic field assumes the form

$$T^{\alpha\beta} = (\rho c^2 + \rho e + p + \mu h^2)U^\alpha U^\beta + (p + \mu h^2/2)g^{\alpha\beta} - \mu h^\alpha h^\beta,$$

which is symmetric and satisfies the invariant conservation law through Einstein's field equation.

Here p, ρ, e and h respectively represent the fluid pressure, the particle density of the matter, the internal energy density and the intensity of the magnetic field where

$$|h|^2 = h_\alpha h^\alpha > 0.$$

Taking the one-fluid approximation (no Hall effect, electron pressure gradient) and neglecting the dissipative processes and spin effects, the basic equations governing the flow of a relativistic thermodynamically perfect magnetofluid of infinite electrical conductivity and constant magnetic permeability μ are (Lichnerowicz 1967)

$$\nabla_\alpha(\rho U^\alpha) = 0, \tag{2.1}$$

$$\nabla_\alpha T^{\alpha\beta} = 0, \tag{2.2}$$

$$\nabla_\beta(U^\alpha h^\beta - U^\beta h^\alpha) = 0, \qquad U^\alpha h_\alpha = 0, \tag{2.3}$$

where

$$T^{\alpha\beta} = \omega U^\alpha U^\beta + (p + \mu/2h^2)S^{\alpha\beta} - \mu h^\alpha h^\beta,$$

$$\omega = \rho c^2(1 + e/c^2 + \mu h^2/2\rho c^2).$$

Equations (2.1)–(2.3) yield the following equations:

$$\rho\sigma c^2 D_U U^\alpha + \nabla_\beta(p + \mu/2h^2)S^{\alpha\beta} - \mu U^\alpha U_\gamma h^\beta \nabla_\beta h^\gamma - \mu h^\beta \nabla_\beta h^\alpha - \mu h^\alpha \nabla_\beta h^\beta = 0, \tag{2.4}$$

$$D_U \eta = 0, \tag{2.5}$$

$$U_\alpha D_U h^\alpha + \nabla_\beta h^\beta = 0, \tag{2.6}$$

$$\tfrac{1}{2} D_U h^2 + h^2 \nabla_\beta U^\beta - h_\alpha h^\beta \nabla_\beta U^\alpha = 0, \tag{2.7}$$

$$c^2 \rho f h_\alpha D_U U^\alpha + h^\alpha \nabla_\alpha p = 0, \tag{2.8}$$

$$U_\alpha \nabla_\beta h^\alpha + h_\alpha \nabla_\beta U^\alpha = 0, \tag{2.9}$$

where

$$\sigma = f + \mu h^2/\rho c^2, \qquad f = 1 + i/c^2.$$

Here i, η and f respectively represent the specific enthalpy, the entropy per unit mass in the instantaneous rest frame and the index of the fluid (Lichnerowicz 1970).

In view of the thermodynamical relation

$$c^2 D_U f = T D_U \eta + (1/\rho) D_U p,$$

equation (2.5) assumes the form

$$D_U p + a^2 \rho \nabla_\alpha U^\alpha = 0 \tag{2.10}$$

where

$$a^2 = \gamma p/\rho, \qquad \gamma = c_p/c_v.$$

3. Jump discontinuities

Let $\Sigma(x^\mu)$ be a time-like regular hypersurface which propagates in the space V_4, and thus separates the space–time into two subregions in either of which the fundamental equations have a continuous solution, but there occurs a strong discontinuity across the wave surface $\Sigma(x^\mu)$. Such a surface across which the flow parameters undergo finite jumps is called a shock wave.

Let a quantity Z, if evaluated upstream (downstream) from the shock surface, be denoted by $Z_1(Z)$. Let $[Z]$ denote the jump in the quantity enclosed as it crosses the shock surface. Then the jump conditions expressing the values of the flow variables just behind in terms of those just ahead of the shock surface are (Lichnerowicz 1970)

$$[\rho U^\alpha] N_\alpha = 0, \tag{3.1}$$

$$[T^{\alpha\beta}] N_\alpha = 0, \tag{3.2}$$

$$[U^\alpha h^\beta - U^\beta h^\alpha] N_\alpha = 0, \tag{3.3}$$

where N_α are the components of the unit four-vector normal to the surface such that

$$N^\alpha N_\alpha = 1.$$

When the magnetic field acts transversely to the direction of propagation of the surface, $h^\alpha N_\alpha = 0$, and hence from (3.1)–(3.3) we get

$$[p] = \frac{\{(\delta+2)p_1\gamma + \rho_1 c^2(1+\delta)(\gamma-1)\}\delta V_1^2 - (\mu/2)(\gamma-1)(1+\delta)^2(\delta^2+2\delta)h_1^2}{(\gamma-1)(1+\delta)^2 + \gamma V_1^2}, \tag{3.4}$$

$$[U^\beta] = \frac{\{[p] + (\mu/2)(\delta^2+2\delta)h_1^2\}\{U_1^\beta - V_1 N^\beta\}}{(1+\delta)\{\rho_1\sigma_1 c^2 - [p] - (\mu/2)(\delta^2+2\delta)h_1^2\}} - \frac{\delta U_1^\beta}{1+\delta}, \tag{3.5}$$

$$[h^\beta] = \delta h_1^\beta, \tag{3.6}$$

$$[h^2] = (\delta^2+2\delta)h_1^2, \tag{3.7}$$

$$[\rho] = \delta\rho_1, \tag{3.8}$$

where $U^\alpha N_\alpha = V$ and δ is the density strength of the shock, which is given by

$$\{V_1^2 + (1+\delta)^2\}\{[p] + (\mu/2)(\delta^2+2\delta)h_1^2\}\{[p] + (\mu/2)(\delta^2+2\delta)h_1^2 - 2\rho_1\sigma_1 V_1^2 c^2\} + (\rho_1\sigma_1 V_1^2 c^2)^2 = 0. \tag{3.9}$$

Equations (3.4)–(3.9) determine the shock discontinuities explicitly in terms of the shock parameters and the flow parameters just ahead of the shock wavefront.

4. Determination of the flow gradients at the rear of the shock

The geometrical and kinematical compatibility conditions of first order of a time-like regular hypersurface of the space V_4 are (Maugin 1976)

$$[\nabla_\alpha Z]=[D_N Z]N_\alpha+g_{\alpha\beta}[D_T^\beta Z], \tag{4.1}$$

$$[D_U Z]=V[D_N Z]+[U_\beta D_T^\beta Z], \tag{4.2}$$

where

$$D_U\equiv U^\alpha\nabla_\alpha, \qquad D_N\equiv N^\alpha\nabla_\alpha,$$

$$D_T^\alpha Z\equiv g^{\alpha\beta}\nabla_\beta Z-N^\alpha D_N Z=a^{ij}x^\alpha_{;i}Z_{;j}.$$

Here $D_N Z$ and $D_T^\alpha Z$ represent the covariant derivative of Z along the normal and the tangent to the shock respectively; $U_\alpha D_T^\alpha Z$ is the generalised form of the δ_t derivative of Thomas (1957); a^{ij} is the first fundamental form of the metric tensor of the shock surface; a semicolon followed by a Latin index denotes covariant differentiation with respect to the corresponding parametric coordinate of the shock surface. We assume, for simplicity, that the flow ahead of the shock is uniform and known. Keeping in mind the aforementioned assumptions and taking jumps in (2.1), (2.3), (2.4), (2.7), (2.8) and (2.10) with the help of (4.1) and (4.2), we obtain

$$VD_N\rho+U_\alpha D_T^\alpha\rho+\rho N_\alpha D_N U^\alpha+\rho g_{\alpha\beta}D_T^\beta U^\alpha=0, \tag{4.3}$$

$$\begin{aligned}VD_N h^\alpha&+U_\beta D_T^\beta U^\alpha+h^\alpha N_\beta D_N U^\beta-U^\alpha N_\beta D_N h^\beta+h^\alpha g_{\nu\beta}D_T^\nu U^\beta\\&-h^\beta g_{\nu\beta}D_T^\nu U^\alpha-U^\alpha g_{\nu\beta}D_T^\nu h^\beta=0,\end{aligned} \tag{4.4}$$

$$\begin{aligned}c^2\rho\sigma VD_N U^\alpha&+c^2\rho\sigma U_\beta D_T^\beta U^\alpha+D_N pS^{\alpha\beta}N_\beta+g_{\nu\beta}D_T^\nu pS^{\alpha\beta}+(\mu/2)D_N h^2S^{\alpha\beta}N_\beta\\&+(\mu/2)g_{\nu\beta}D_T^\nu h^2S^{\alpha\beta}-\mu h^\alpha N_\beta D_N h^\beta-\mu U^\alpha U_\gamma h^\beta g_{\nu\beta}D_T^\nu h^\gamma\\&-\mu h^\beta g_{\nu\beta}D_T^\nu h^\alpha-\mu h^\alpha g_{\nu\beta}D_T^\nu h^\beta=0,\end{aligned} \tag{4.5}$$

$$(\mu/2)VD_N h^2+(\mu/2)U_\alpha D_T^\alpha h^2+\mu h^2N_\alpha D_N U^\alpha+\mu h^2g_{\nu\beta}D_T^\nu U^\beta-\mu h_\alpha h^\beta g_{\nu\beta}D_T^\nu U^\alpha=0, \tag{4.6}$$

$$c^2\rho fVU_\alpha D_N h^\alpha+c^2\rho fU_\alpha U_\beta D_T^\beta h^\alpha-h^\beta g_{\nu\beta}D_T^\nu p=0, \tag{4.7}$$

$$VD_N p+U_\alpha D_T^\alpha p+\rho a^2N_\alpha D_N U^\alpha+\rho a^2g_{\nu\beta}D_T^\nu U^\beta=0. \tag{4.8}$$

Contracting (4.5) with N_α, we get

$$\begin{aligned}c^2\rho\sigma VN_\alpha D_N U^\alpha&+(1+V^2)D_N p+(\mu/2)(1+V^2)D_N h^2+c^2\rho\sigma N_\alpha U_\beta D_T^\beta U^\alpha\\&+VU_\beta D_T^\beta p+(\mu/2)VU_\beta D_T^\beta h^2-\mu VU_\gamma h_\beta D_T^\beta h^\gamma\\&-\mu N_\alpha h_\beta D_T^\beta h^\alpha=0.\end{aligned} \tag{4.9}$$

Eliminating $D_N h^2$ from (4.6) and (4.9), we obtain

$$\begin{aligned}\{c^2\rho\sigma V^2&-\mu h^2(1+V^2)\}N_\alpha D_N U^\alpha+V(1+V^2)D_N p\\&+c^2\rho\sigma VN_\alpha U_\beta D_T^\beta U^\alpha-(\mu/2)(1+V^2)U_\beta d_T^\beta h^2\\&+V^2U_\beta D_T^\beta p+(\mu/2)V^2U_\beta D_T^\beta h^2-\mu VN_\alpha h_\beta D_T^\beta h^\alpha-\mu V^2U_\gamma h_\beta D_T^\beta h^\gamma\\&-\mu h^2(1+V^2)g_{\nu\beta}D_T^\nu U^\beta+\mu h_\alpha(1+V^2)h_\beta D_T^\beta U^\alpha=0.\end{aligned} \tag{4.10}$$

Eliminating $N_\alpha D_N U^\alpha$ from (4.8) and (4.10), we get

$$D_N p = V^{-1}\{V^2(c^2\sigma - a_e^2) - a_e^2\}^{-1}\{\rho\sigma V a^2 c^2 N_\alpha U_\beta D_T^\beta U^\alpha + a^2 V^2 U_\beta D_T^\beta p + (\mu/2) V^2 a^2 U_\beta D_T^\beta h^2 - 2\mu V^2 a^2 U_\gamma h_\beta D_T^\beta h^\gamma - \mu a^2 U_\alpha h_\beta D_T^\beta h^\alpha - \mu a^2 V N_\alpha h_\beta D_T^\beta U^\alpha - c^2 a^2 V^2 \rho\sigma g_{\nu\beta} D_T^\nu U^\beta - (\mu/2) a^2(1+V^2) U_\alpha D_T^\alpha h^2 - \sigma c^2 V^2 U_\alpha D_T^\alpha p + \mu h^2(1+V^2) U_\alpha D_T^\alpha p\}, \tag{4.11}$$

where $a_e^2 = a^2 + \mu h^2/\rho$ is the effective velocity of sound.

From (4.4), (4.7) and (4.8), we obtain

$$D_N h^\alpha = \{c^2\rho f a^2 V\}^{-1}\{c^2 f(V h^\alpha D_N p + h^\alpha U_\beta D_T^\beta p + \rho a^2 h_\beta D_T^\beta U^\alpha - \rho a^2 U_\beta D_T^\beta h^\alpha) - a^2 U^\alpha h_\beta D_T^\beta p\}. \tag{4.12}$$

In view of (4.3), we have

$$D_N \rho = -V^{-1}\{\rho N_\alpha D_N U^\alpha + U_\alpha D_T^\alpha p + \rho g_{\alpha\beta} D_T^\alpha U^\beta\}. \tag{4.13}$$

Elimination of $N_\alpha D_N U^\alpha$ from (4.6) and (4.8) yields

$$D_N h^2 = \{\rho a^2 V\}^{-1}\{2 V h^2 D_N p + 2h^2 U_\alpha D_T^\alpha p - \rho a^2 U_\alpha D_T^\alpha h^2 + 2\rho a^2 h_\alpha h_\beta D_T^\beta U^\alpha\}. \tag{4.14}$$

From equation (4.5), we get

$$D_N U^\alpha = \{c^2\rho\sigma V\}^{-1}\{\mu h^\alpha N_\beta D_N h^\beta - (D_N p + \mu/2 D_N h^2) S^{\alpha\beta} N_\beta - c^2\rho\sigma U_\beta D_T^\beta U^\alpha - S^{\alpha\beta} g_{\nu\beta} D_T^\nu (p + \mu/2 h^2) + \mu U^\alpha U_\gamma h_\beta D_T^\beta h^\gamma + \mu h_\beta D_T^\beta h^\alpha + \mu h^\alpha g_{\nu\beta} D_T^\nu h^\beta\}. \tag{4.15}$$

Equations (4.11)–(4.15) show that the necessary and sufficient conditions for the existence and uniqueness of flow behind a shock wave are given by

$$V \neq 0, \qquad V^2(c^2\sigma - a_e^2) - a_e^2 \neq 0.$$

The first condition $V \neq 0$ suggests that the discontinuity appearing in the flow under consideration is not a tangential discontinuity, whereas the second condition $\{V^2(c^2\sigma - a_e^2) - a_e^2\} \neq 0$ suggests that the discontinuity under consideration is not a sonic discontinuity. This implies that a shock wave is neither a tangential discontinuity nor a sonic discontinuity/weak wave.

The surface derivative and δ_t derivative of N^α and V are obtained (Maugin 1976) in the following form:

$$N^\alpha_{;k} = -b_{km} a^{im} x^\alpha_{;i} = -a^{im} x^\nu_{;km} x^\alpha_{;i} N_\nu, \tag{4.16}$$

$$U_\beta D_T^\beta N^\alpha = -b^{ij} U_\beta x^\beta_{;i} x^\alpha_{;j}, \tag{4.17}$$

$$U_\beta D_T^\beta V = a^{ij} U_\beta x^\beta_{;i} V_{;j}, \tag{4.18}$$

where b_{ij} are covariant components of the second fundamental tensor of the shock surface.

Applying the operator $U_\beta D_T^\beta$ on (3.4)–(3.8) and using (4.17) and (4.18), we can obtain expressions for $U_\beta D_T^\beta U^\alpha$, $U_\beta D_T^\beta h^\alpha$, $U_\beta D_T^\beta h^2$, $U_\beta D_T^\beta p$ and $U_\beta D_T^\beta \rho$. Similarly, expressions for $D_T^\beta U^\alpha$, $D_T^\beta h^\alpha$, $D_T h^2$, $D_T^\beta p$ and $D_T^\beta \rho$ can be obtained. Thus the above quantities involved in the expressions for gradients are expressible in terms of known quantities. Such quantities depend upon the shape, speed and strength of the shock and the flow and field parameters just ahead of the shock.

5. Determination of vorticity and current density

The components of vorticity and current density are given by

$$W^{\alpha}=\tfrac{1}{2}\eta^{\alpha\beta\gamma\delta}_{\nabla_{\gamma}U_{\beta}U_{\delta}}, \tag{5.1}$$

$$J^{\alpha}=\tfrac{1}{2}\eta^{\alpha\beta\gamma\delta}_{\nabla_{\beta}(U_{\gamma}h_{\delta})}, \tag{5.2}$$

where $\eta^{\alpha\beta\gamma\delta}=(-g)^{1/2}\varepsilon^{\alpha\beta\gamma\delta}$, $\varepsilon^{\alpha\beta\gamma\delta}$ being the four-dimensional permutation tensor.

Taking jumps of (5.1) and (5.2), we get

$$[W^{\alpha}]=\tfrac{1}{2}\eta^{\alpha\beta\gamma\delta}_{(D_NU_{\beta}N_{\gamma}U_{\delta}+g_{\nu\gamma}D^{\nu}_{T}U_{\beta}U_{\delta})}, \tag{5.3}$$

$$[J^{\alpha}]=\eta^{\alpha\beta\gamma\delta}_{(D_NU_{\gamma}N_{\beta}h_{\delta}+D_Nh_{\delta}N_{\beta}U_{\gamma}+g_{\nu\beta}D^{\nu}_{T}U_{\gamma}h_{\delta}+g_{\nu\beta}D^{\nu}_{T}h_{\delta}U_{\gamma})}. \tag{5.4}$$

Contracting (5.3) and (5.4) with N_{α}, we obtain

$$W^{\alpha}N_{\alpha}=\tfrac{1}{2}\eta^{\alpha\beta\gamma\delta}_{U_{\delta}N_{\alpha}g_{\nu\gamma}D^{\nu}_{T}U_{\beta}}, \tag{5.5}$$

$$J^{\alpha}N_{\alpha}=\eta^{\alpha\beta\gamma\delta}_{(h_{\delta}g_{\nu\beta}D^{\nu}_{T}U_{\gamma}+U_{\gamma}g_{\nu\beta}D^{\nu}_{T}h_{\delta})N_{\alpha}}. \tag{5.6}$$

In view of (4.11), (4.12) and (4.15), equations (5.3) and (5.4) indicate that the vorticity and current density generated by the shock can be derived purely from dynamical considerations. A similar result holds in the non-relativistic frame also. Furthermore, contractions of (5.3) and (5.4) with U_{α} yield

$$W^{\alpha}U_{\alpha}=0, \tag{5.7}$$

$$J^{\alpha}U_{\alpha}=-W^{\alpha}h_{\alpha}. \tag{5.8}$$

Equation (5.7) shows that the vorticity generated by the hydromagnetic shock is orthogonal to the world velocity.

Contracting (5.4) with h_{α}, we get

$$J^{\alpha}h_{\alpha}=-2\hat{W}^{\alpha}U_{\alpha} \tag{5.9}$$

where

$$\hat{W}^{\alpha}=\tfrac{1}{2}\eta^{\alpha\beta\gamma\delta}_{h_{\beta,\gamma}h_{\delta}},$$

which has already been defined by Bray (1975) as the magnetic vorticity.

Taking decomposition of the electric current into conduction and convection parts, (5.8) and (5.9) assume, respectively, the forms

$$W^{\alpha}h_{\alpha}=\varepsilon, \tag{5.10}$$

$$\hat{W}^{\alpha}U_{\alpha}=0. \tag{5.11}$$

Equation (5.10), which implies that the charge distribution depends on the magnetic field strength and on the vorticity relative to the local inertial frames, confirms Bekenstein's prediction (Bekenstein and Oron 1979) for the interior magnetohydrodynamic structure of a relativistic neutron star. Equation (5.11) shows that the magnetic vorticity is orthogonal to the world line.

The results derived herein are expected to be of great interest for relativistic astrophysics (Ruderman 1972, Bray 1975, Bekenstein and Oron 1979). They may be useful for the 'seismology' of massive stellar objects in which a relativistic treatment is expected. The interior of neutron stars and pulsars and the gas clouds in galaxies may be regarded as an ideal, infinitely conducting single fluid with a frozen-in magnetic field (Bekenstein and Oron 1979, Ruffini 1975). The appearance of shock waves in such astrophysical objects generates vorticity in the superfluid which mimics the behaviour of a rotating magnetofluid. The differential identities may be useful for investigation of the magnetic structure of a rotating neutron star's interior and the orientation of fields in astrophysical objects like galaxies, pulsars, gravitational collapse, sun spots and spiral arms. Detailed applications of the results to more general situations of astrophysical and cosmological importance are under current investigation and will be published elsewhere.

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